CS361B: Homework 3

Due date: June 3, 2014 at 12:15PM

Maximum Multicimmodity Flow: We are given a graph $G(V, E)$ with capacities $u(e)$ on the edges, and k pairs of terminals $(s_i, t_i), i = 1, 2, \ldots, k$. The goal is to route flow d_i from s_i to t_i, so that $\sum_{i=1}^{k} d_i$ is maximized. (Note that the sets $\{s_i\}$ and $\{t_i\}$ might not be disjoint.) We will develop a combinatorial algorithm for this problem similar to the algorithm for maximum concurrent flow. If $f(P)$ denotes the flow along path P, and Q_i denotes the set of paths from s_i to t_i, we can formulate the following positive linear program:

$$\text{Maximize} \sum_{i=1}^{k} d_i$$

$$\begin{array}{rcll} \sum_{P \in Q_i} f(P) & \geq & d_i & \forall i \\ \sum_{P: e \in P} f(P) & \leq & u(e) & \forall e \in E \\ f(P) & \geq & 0 & \forall P \in \bigcup_{i=1}^{k} Q_i \\ d_i & \geq & 0 & \forall i \end{array}$$

Problem 3-1. Dual Problem: Formulate the dual of this problem. Use the variables $l(e)$ for the edge constraints, and the variables dist_i for the terminal constraints. Define the volume $D(l)$ of the system as $\sum_e l(e)u(e)$.

1. Show that for the optimal solution, the function $l(e)$ is a metric (in the sense that it is non-negative and, for all $x, y, z \in V$ such that $xy \in E$, $l(xy) \leq c_l(xz) + c_l(zy)$, where $c_l(vw)$ denotes the length of the shortest path from v to w when edge lengths are defined by l).
2. Show that for the optimal solution, dist_i can be set to the shortest path length from s_i to t_i under the metric l without changing the value of the optimum.
3. Given a metric l, let $\alpha(l)$ denote the minimum distance between terminal pairs. Show that the dual is effectively minimizing $\frac{D(l)}{\alpha(l)}$ over all length metrics l.
4. Suppose the variables $l(e)$ were constrained to be either 0 or 1. In this case, what problem is the dual program solving?

Problem 3-2. Complementary Slackness: Write down the primal and dual complementary slackness constraints. Consider the optimal primal and dual solutions.

1. Show that for $P \in Q_i$, where Q_i is the set of paths from s_i to t_i, if $f(P) > 0$, then the length of P in metric l is one.
2. Show that if $l(e) > 0$, then edge e is saturated.

Problem 3-3. The Algorithm: We will solve this problem for the case of unit capacities $u(e) = 1$. The algorithm proceeds in iterations. Let l_{i-1} be the length function at the beginning of the i^{th} iteration, and f_{i-1} denote the flow routed so far. Let $\alpha(i-1)$ denote the minimum distance between terminals in metric l_{i-1}, and $D(i-1)$ denote the volume of the system. Let P be a path of length

$\alpha(i-1)$ connecting some terminal pair. We push one unit of flow along P, and for edge $e \in P$, set $l_i(e) = l_{i-1}(e)(1+\epsilon)$. We stop at the first time t such that $\alpha(t) \geq 1$.

Essentially, the algorithm finds the path with minimum capacity violation and pushes one unit of flow along it. This path is the shortest path using a length function which is exponential in the violation. Note that f_t does not satisfy capacity constraints and is therefore infeasible.

Initially, we set $l_0(e) = \delta$ for all edges. We will choose δ later. Let β denote the optimal value of the dual.

Note that $\alpha(0) \leq \delta n$. Also note that $f_i = i$.

1. Show that $D(i) = D(0) + \epsilon \sum_{j=1}^{i} \alpha(j-1)$.
2. Consider the length function $l_i - l_0$, and let $\alpha(l_i - l_0)$ denote the length of the shortest path from any source to the corresponding sink under this length function. Show that $\beta \leq \frac{D(i)-D(0)}{\alpha(l_i-l_0)}$, and conclude that $\alpha(i) \leq \delta n + \frac{D(i)-D(0)}{\beta}$.
3. Now show that $\alpha(i) \leq \delta n(1+\epsilon/\beta)^i$. Conclude that $\alpha(i) \leq \delta n e^{\epsilon i/\beta}$.
4. Finally, show that $f_t = t \geq \frac{\beta \ln(\delta n)^{-1}}{\epsilon}$.

Problem 3-4. **Feasible Flow:** The algorithm described above could easily violate capacities. Note that whenever we route one unit of demand through an edge e, we increase its length by a factor of $1+\epsilon$.

1. Using the fact that $l_0(e) = \delta$, and t is the first time instant for which $\alpha(t) \geq 1$, show that the total flow through e is at most $\log_{1+\epsilon} \frac{1+\epsilon}{\delta}$.
2. Show that $\frac{f_t}{\log_{1+\epsilon} \frac{1+\epsilon}{\delta}}$ is a feasible flow.

Problem 3-5. **Approximation Ratio:** Let γ denote the ratio between the optimal dual solution and the flow we obtain, that is $\gamma = \frac{\beta}{f_t} \log_{1+\epsilon} \frac{1+\epsilon}{\delta}$. Show that for $\delta = (1+\epsilon)((1+\epsilon)n)^{-1/\epsilon}$, $\gamma \leq (1-\epsilon)^{-2}$.

Problem 3-6. **Running Time:** Show that the algorithm described above computes a $(1-\epsilon)^{-2}$ approximation to max multicommodity flow in time $O((\frac{m}{\epsilon^2} \log n) k T_{\text{sp}})$, where T_{sp} is the time taken to compute single source shortest paths.

Problem 3-7. **Optional:** Suppose we remove the unit capacity assumption. We modify the algorithm as follows. As before, let P be the shortest path in metric l_{i-1}. Let u denote the minimum capacity edge along this path. We push u units of flow along this path, and for edge e along this path, set $l_i(e) = l_{i-1}(e)(1+\epsilon u/u(e))$. We terminate at the first time t such that $\alpha(t) \geq 1$. The values $l_0(e)$ are set as before. Note that f_i is no longer equal to i. Show that after appropriate scaling of the flow and choice of δ, this algorithm produces a $(1-\epsilon)^{-2}$ approximation to maximum multicommodity flow.